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ACOUSTICAL MODELLING OF THE SURFACE SOURCES — IV INTERPOLATION MODEL, AXISYMMETRIC PROBLEM

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In the paper, the quadratic shape function were applied to construct an interpolation model of a plane axisymmetric source. Optimizing the least squares distance between the exact directivity function and directivity of the model, an optimal interpolation model was found. The optimal model was considered as a function of the shape of the vibration velocity, wave number and the place of the driving surface in the baffle. An optimal model was compared to other interpolation models, namely regular and irregular ones.

1. Introduction

An exact solution of the radiation acoustic problem of a complicated shape source and arbitrary vibration velocity can be obtained by numerical methods only. This problem is solved applying the boundary element method (BEM). The first step of BEM is discretization of the geometry of the source and the vibration velocity.

In the classical BEM the surface of the source is replaced by planar elements. In addition, the vibration velocity is represented by piecewise constant values on these elements. Then the elements vibrate as pistons. In this way a piston model of the source (PM) is obtained [15]. This model well approximates the source if it contain a lot of elements.

BEM was improved applying quadratic shape functions (interpolating functions) [1, 14]. Because the same shape functions are usually used to approximate both the geometry of the source and the vibration velocity [9, 13, 15–18, 21], then the source is replaced by an array of isoparametric elements. This array is called interpolation model (IM). The IM model yields more accurate result than PM model for the same number of surface elements. If the IM model is solved numerically [7, 10, 12], the disadvantage is arisen namely it is necessity to calculate double singular integrals. However, in the case of PM, double integrals can be converted analytically to single integrals [9].

Recently, an excellent computer program called BEMAP [19] was worked out to solve the acoustic field of the IM of an arbitrary source.

In Refs. [2, 3, 11], a PM of rectangular membranes was taken into account where the constant vibration velocity was assumed on each element. In paper [2], only the first mode of the membrane was analyzed but in [11] higher modes were dealt with. General rules of modelling plane rectangular sources with an infinite and rigid baffle were given in [3]. Axisymmetric sources with an infinite and rigid baffle were considered in [8].

It has been shown in references that the primary discretization (gives primary elements) is imposed by the singularities of geometry (edges and corners) and singularities of vibration velocity (nodal lines and extremum). Such obtained model did not give sufficient results. Then, in all the above mentioned works, more refined regular discretization (secondary discretization giving secondary elements) of primary elements was applied. But, as mentioned above, such a model (i.e. PM) consists of too many element or, during numerical calculations for IM, singular double integrals appear.

As pointed out in [4, 5, 6], the number of surface elements in PM may be reduced applying irregular secondary discretization. In particular an optimal piston model (PM_o) may be obtained. This model approximates most accurately the source for a fixed number of elements and fixed method for calculation of the vibration velocities on elements.

This paper discusses the application of irregular secondary discretization for constructing an optimal interpolation model (IM_o). As an example, an axisymmetric

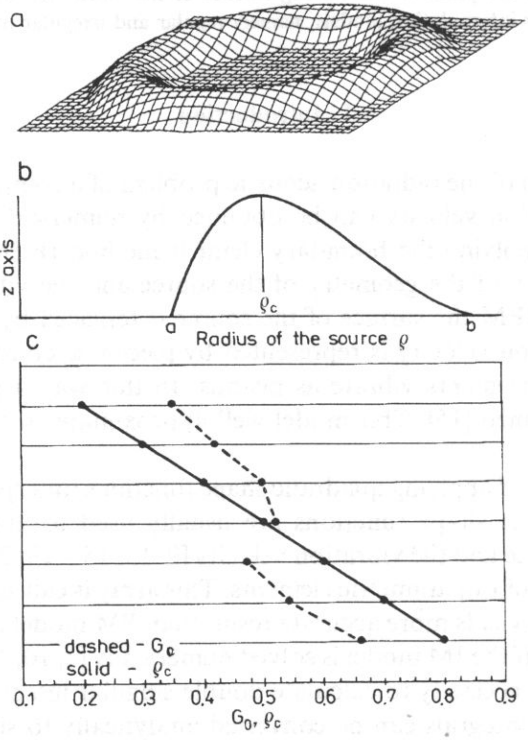

Fig. 1. (a) Picture of the driving surface, (b) Cross-section of the driving surface, (c) Places ρ_c of the maximum of vibration velocity and boundary G_o of the optimal interpolation model.

source with a plane infinite baffle was considered. Due to a symmetry with regard to z-axis, only the cross-section line of vibration velocity needs to be discretized. However, it is assumed that this line is asymmetrical with regard to its nodal points (see Fig. 1).

The problem of optimal model was analyzed depend on asymmetry of the line of vibration velocity, wave number (frequency of vibration) and the place of the driving surface in the baffle. Searching of IM_o quite similar as in Ref. [6], a least squares distance between exact directivity function and of the model was optimalized. In the paper, it will not put any emphasis on mathematical aspects of optimalization problem. Rather, the numerical problems are presented with some details.

1. Directivity function of axisymmetric source

The acoustic field of a circular driving surface set in an infinite and rigid baffle is considered. Assuming axisymmetric (AS) vibration velocity, an axially symmetric field is obtained. The acoustic potential of this acoustic field is given by Helmholtz–Rayleigh integral [5]. The directivity function of such field normalized by the field of point source placed in an origin of coordinates is given by [6]:

$$Q_{\mathrm{ON}}(\gamma)=\int_{\rho} v_{\mathrm{ON}}(\rho)\, J_o(k\rho \sin\gamma)\, \rho d\rho , \tag{1.1}$$

where $J_o(x)$ — Bessel function of first kind and zero order, k — wave number, ρ, ϕ — polar coordinates, γ — azimuthal angle in spherical coordinates, $v_{\mathrm{ON}}(\rho)$ — vibration velocity function, given explicite by

$$v_{\mathrm{ON}}(\rho)=C_1 \sin\left(\pi\frac{\rho-a}{b-a}\right)\exp\left(-C_2\frac{\rho-a}{b-a}\right). \tag{1.2}$$

The coefficients C_1, C_2 have been chosen to attain the assumed asymmetry of $v_{\mathrm{ON}}(\rho)$. The Eq. (1.1) has been described extensively in Refs. [4, 5].

2. Directivity function of the model

After discretization of the driving surface, the vibration velocity of arbitrary j-element can be expressed in terms of its nodal values which are chosen in i-points as follow [1]:

$$v_j(\rho) \cong \sum_{i=1}^{I} N_i(\rho)\, v_j(\rho_i) , \tag{2.1}$$

where $N_i(\rho)$ — shape functions.

Equation (2.1) presents interpolation of function $v_j(\rho)$. From the properties of this interpolation it follows that the right hand side equals the left hand one only at i-points. In general, except i-points, the sign of equality cannot be placed in Eq. (2.1). An absolute error, as an effect of equality in Eq. (2.1) can be calculated analytically. But it will be an error of vibration velocity which, in acoustic, has no useful physical meaning.

The error brought about by Eq. (2.1) was evaluated indirectly analysing the convergence of both the exact directivity function and the model. The vibration velocity of this model was expressed by the right hand side of Eq. (2.1). The least squares distance was assumed as the measure of convergence of directivity functions [5]. Its value may be interpreted as an interpolation model error (IME). Substituting (2.1) into (1.1) we obtain

$$Q_{\mathrm{ON}}(\gamma)=\sum_{j=1}^{J}\sum_{i=1}^{I} v_j(\rho_i)\int_{\rho_j} N_i(\rho)\,J_o(k\rho\sin\gamma)\,\rho d\rho\,. \tag{2.2}$$

Using formula (2.2) in numerical calculation is somewhat difficult because of the boundaries of each j-element change. To overcome this difficulty the shape function, coordinate ρ and vibration velocity are expressed in nondimensional coordinates. In literature e.g. [14, 20], quadratic shape functions are recommended to interpolate a function of lower order

$$\begin{aligned} N_1(\xi)&=\xi(\xi-1)/2,\\ N_2(\xi)&=1-\xi^2,\\ N_3(\xi)&=\xi(\xi+1)/2, \end{aligned} \tag{2.3}$$

where $\xi\in\langle -1,1\rangle$.

In nondimensional coordinates the coordinate ρ is given by

$$\rho(\xi)=\sum_{i=1}^{3} N_i(\xi)\,\rho_i, \tag{2.4}$$

Substituting (5) into (6) and assuming

$$\rho_1=G_1,\qquad \rho_2=\frac{1}{2}(G_1+G_2),\qquad \rho_3=G_2, \tag{2.5}$$

leads to

$$\rho(\xi)=\frac{1}{2}[\xi(G_2-G_1)+G_2+G_1]. \tag{2.6}$$

In Eq. (2.5) G_1 and G_2 are boundaries of j-element. These boundaries are consequence of discretization.

Now the values of ρ_i can be calculated from Eq. (2.6) for $\xi=\xi_i$ where $\xi_1=-1$, $\xi=0$, $\xi_3=1$.

In Eq. (2.2), it is advisable to express also $v_j(\rho_i)$ as a function of ξ_i. Using Eq. (2.4) we obtain

$$v_j(\rho_i)=v_j[\rho(\xi_i)]\,. \tag{2.7}$$

Because the integration is to do with respect ξ instead of ρ than the Jacobian J_B of the transformation (2.4) is

$$J_B=\frac{\partial\rho}{\partial\xi}=\frac{1}{2}(G_2-G_1)\,. \tag{2.8}$$

Making use of Eqs. (2.3), (2.4), (2.7) and (2.8) in Eq. (2.2) gives

$$Q_{\mathrm{ON}}(\gamma)=\sum_{j=1}^{J}\sum_{i=1}^{3}v_j[\rho(\xi_i)]\int_{-1}^{1}N_i(\xi)\,J_o[k\rho(\xi)\sin\gamma]\,\rho(\xi)\,J_B(\xi)\mathrm{d}\xi\,. \tag{2.9}$$

Equation (2.9) presents the directivity function of the interpolation model and together with (1.1) constitutes the basis of numerical calculations.

3. Numerical calculations

In the numerical calculations a vibration velocity function without internal points is applied. Furthermore, its maximum is not half distance between nodal points [5, 6], it reaches its maximum value at ρ_c ($\rho_c\in\langle a,\ b\rangle$). Outer and inner radius of the driving surface are marked by a and b respectively, $b=a+1$ (Fig. 1).

The numerical calculations are performed for three two-element models, i.e. $J=2$, obtained as a result of:

1 — regular discretization leading to the regular interpolation model (IM_r),

2 — irregular discretization leading to the irregular interpolation model (IM_i) where the discretization passes on ρ_c, i.e. the maximum of the vibration velocity,

3 — optimization of least squares distance between exact directivity function and the directivity function of optimal interpolation model (IM_o).

The boundary between the elements of the over determined models was denoted as $G_r=G_{0.5}=0.5$, $G_i=\rho_c$ and G_o respectively. The models were compared with one another based on the analysis of the least squares distance among the exact directivity functions exact and of the models.

This comparison is made in relation to:

A — shape of vibration velocity function given by ρ_c

B — wave number k,

C — position of driving surface in the baffle given by a.

Here, least squares distance is a measure of the error of each of the interpolation models (IME) [6]. This error depends on both the discretization (number of elements and boundaries among them) and the number of i-points chosen on the element.

I. In first part of the numerical calculations, the influence of the shape vibration velocity on the construction of the model and on their directivity functions was searched; it was assumed that $k=2.5$, $a=0$. Boundary G_o if IM_o (dashed line) and the place ρ_c of maximum vibration velocity (solid line) were shown in Fig. 1c. An examination of this figure indicates that boundary G_o does not coincide with maximum ρ_c. Only for $\rho_c=.5$ boundary G_o is quite close to this value i.e. $G_o=.52$.

The errors of the models as a function of shape vibration velocity are presented in Fig. 2. It can be seen from this figure that IM_o assures the least error IM_oE. Furthermore, in the range $\rho_c \in \langle .2, .6 \rangle$ this error is almost constant. All models give a good convergence only for $\rho_c \cong .5$.

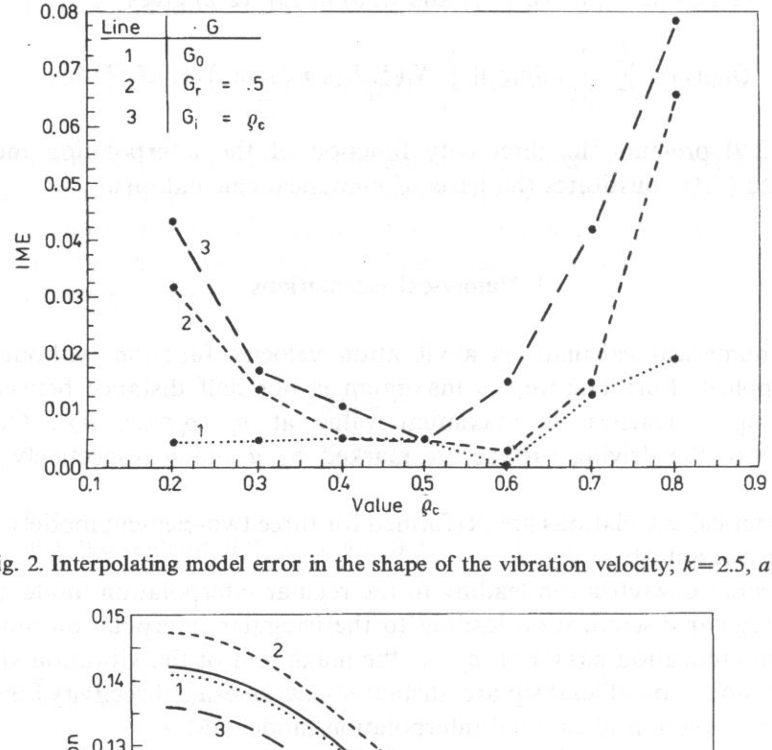

Fig. 2. Interpolating model error in the shape of the vibration velocity; $k=2.5$, $a=0$.

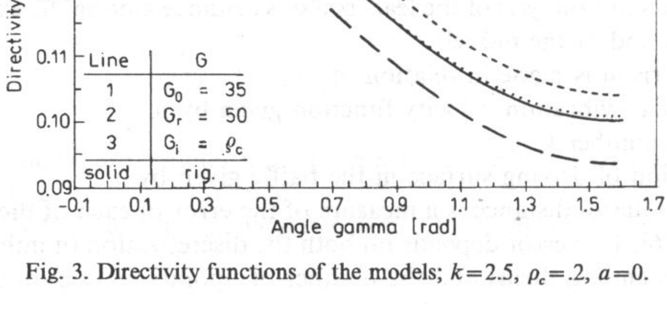

Fig. 3. Directivity functions of the models; $k=2.5$, $\rho_c=.2$, $a=0$.

To check the convergence of exact directivity function and of the model, expressed by IME, the directivities of each of the models, for $\rho_c = .2$, were depicted in Fig. 3. Of course, the best agreement of the directivities is for IM_o. While examining Fig. 3 it is interesting to note that, for $\rho_c = .2$, $IM_oE = .004439$ and it is about 10 times less than $IM_iE = .043190$, but about 7 times less than $IM_{0.5}E = .031444$.

II. In the second part of the numerical calculations the influence of the wave number k on the models is investigated. The numerical calculations are run only for one shape of the vibration velocity function i.e. $\rho_c = .2$ and $a = 0$ is assumed. Values of G_o, for separate k, are given in Table 1.

Table 1.

k	1	2	3	4	5	6	7	8	9	10	20
G_o	.37	.36	.35	.34	.34	.34	.34	.34	.33	.33	.32

As can be seen from Table 1, boundary G_o does not significantly change for $k \in \langle 1, 20 \rangle$, particularly for its higher values. Figure 4 presents the values of the model errors as a function of k. This figure shows that the error IM_0E is the least and its changeability is not large.

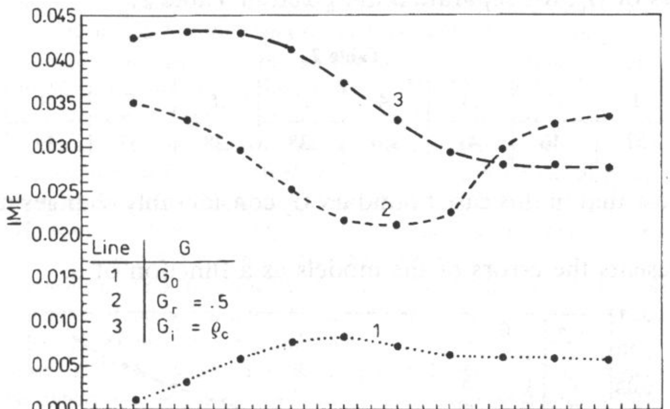

Fig. 4. Interpolating model error in the wave number k; $\rho_c = .2$, $a = 0$.

The discrepancy between the exact directivity function and that of the model, for $k = 10$, i.e. such directivity functions which have sharp minimum, is given in Fig. 5. Numerical calculations performed in detail pointed out that the sharp minimum of exact directivity function and of the models IM_o, IM_r, IM_i are in space described by angle γ equal 0.8, 0.8, 0.73 and 0.83 respectively. The discrepancy between directivity functions presented above reflects the errors given in Fig. 4 for $k = 10$; $IM_oE = .005568$ and it is about 5 times less than $IM_iE = 0.0276395$ and about 6 times less than $IM_rE = 0.3342$.

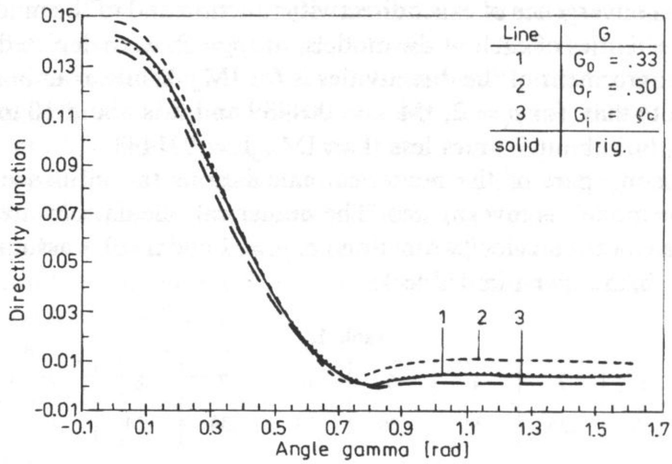

Fig. 5. Directivity functions of the models; $k=10$, $\rho_c=.2$, $a=0$.

III. In the last part of numerical calculations, the influence of the place of driving surface described by value a on the models is considered; $\rho_c=0.2$ and $k=2.5$ are assumed. Values of G_o, for separate a are given in Table 2.

Table 2.

a	0.	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
G_o	.35	.37	.40	.41	.40	.38	.38	.37	.37	.37	.37

The data show that in this case boundary G_o considerably changes for little values a only.

Figure 6 presents the errors of the models as a function of a.

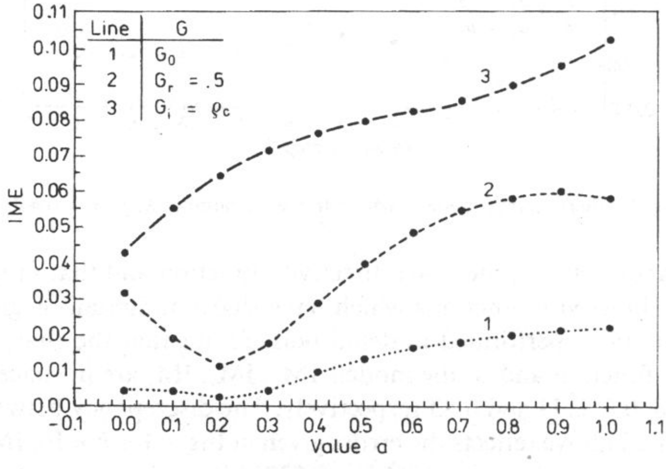

Fig. 6. Inteprolating model error in the place of driving surface in the baffle; $k=2.5$, $\rho_c=.2$.

4. Conclusions

The numerical analysis performed in this paper results in noticeable conclusions:

1 — directivity function of optimal interpolation model is more convergent to exact directivity than the directivity of a regular model and irregular model if the discretization passes on the maximum of vibration velocity,

2 — the best convergence directivity function of IM_o and exact directivity is noted for:

- vibration velocity symmetric to its nodal points,
- low values of wave number,
- the place of the vibrating surface near the axis of the source.

3 — the place of G_o depends a low degree on the wave number k.

References

[1] J.K. Bathe, E.L. Wilson, *Numerical methods in finite element analysis*, Prentice Hall, Englewood Cliffs 1976.

[2] A. Brański, R. Panuszka, M. Zabawa, *Sound radiation by a rectangular plate represented by a model in form of a system of rectangular pistons*, Proceedings Noise Control'79, Vol. II, 53–56, Warszawa 1979.

[3] A. Brański, *Acoustical model of the plates* (in Polish), WSP, Rzeszów 1991.

[4] A. Brański, *Acoustical modelling of the surface sources — I. Piston model, discretizating error, axisymmetric problem*, Archives of Acoustics, **20**, 4, 311–326 (1995).

[5] A. Brański, *Acoustical modelling of the surface sources — II. Piston model, symmetrical vibration velocity, discretizating error, axisymmetric problem*, Archives of Acoustics, **20**, 4, 327–345 (1995).

[6] A. Brański, *Acoustical modelling of the surface sources — III. Piston model, model error, axisymmetric problem*, Archives of Acoustics, **20**, 4, 347–359 (1995).

[7] C.C. Chien, H. Rajiyah, S.N. Atluri, *An effective method or solving the hypersingular integral equations in 3-D acoustic*, J.A.S.A., **88**, 2, 918–937 (1990).

[8] A. Dobrucki, *Modelling of the Acoustical Systems* (in Polish), Raport nr I-28 (s-54) 90, Wrocław 1990.

[9] Y. Furue, *Calculation of sound diffraction by means of the integral equation*. The second Joint Meeting of ASA and ASJ, (1988), private communication.

[10] G. Krishnasamy, L.W. Schmerr, T.J. Rudolphi, F.J. Rizzo, *Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering*, Trans. ASME J. Appl. Mech., **57**, 404–414 (1990).

[11] L. Leniowska, A. Brański, *The directivity function of the model rectangular plate vibrating with the higher modes*, Proceedings Noise Control'88, Vol. I, 263–266, Cracow 1988.

[12] P. Linz, *On the approximate computation of certain strongly singular integrals*, Computing 35, 345–353 (1985).

[13] W.L. Meyer, W.A. Bell, B.T. Zinn, *Boundary integral soluton of three dimensional acoustic radiation problems*, J. Sound Vib., **59**, 2, 245–262 (1978).

[14] A. Ralson, *A first course in numerical analysis* (in Polish), PWN, Warszawa, 1983.

[15] A.F. Seybert, B. Soenarko, F.J. Rizzo, D.J. Shippy, *Application of the BIE method to sound radiation problems using an isoparametric element*, ASME Trans., **106**, 414–420 (1984).

[16] A.F. Seybert, B. Soenarko, F.J. Rizzo, D.J. Shippy, *An advanced computational method for radiation and scattering of acoustic waves in three dimensions*, J.A.S.A., **77**, 2, 362–368 (1985).

[17] A.F. Seybert, B. Soenarko, *Radiation and scattering of acoustic waves from bodies of arbitrary shape in a three-dimensional half space*, ASME Trans., J. Vib. Acoust. Stress Rel. Dsgn., **110**, 112–117 (1988).

[18] A.F. Seybert, T.W. Wu, *Modified Helmholtz integral equation for bodies sitting on the infinite plane,* J.A.S.A., **85**, 1, 19–23 (1989).

[19] A.F. Seybert, T.W. Wu, G.C. Wan, *BEMAP-Computer Program* Spectronics, INC., 2100 Elgin Place, Lexington, KY 40515, private communication.

[20] A.F. Seybert, Z.H. Jia, T.W. Wu, *Solving knife-edge scattering problems using singular boundary elements,* J.A.S.A., **91**, (3), 1278–1283 (1992).

[21] T. Terai, *On calculation of sound fields around three dimensional object by integral equation methods,* J.S.V., **69**, 1, 71–100 (1980).